AL-BĪRŪNĪ ON INDIAN ARITHMETIC

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Al-Bīrūnī (A.D. 973-1048), the great Islamic encyclopaedist stayed in India from A.D. 1017 to A.D. 1030 and studied critically the arts, literature and sciences of the Indians. His observations on some of the topics and concepts of Indian arithmetic namely numerals, decimal place-value, *pāṭī* and the use of dust numerals, Indian arithmetic in Arabia, rule of three and fraction have been analysed in this paper to show the type of arithmetical elements and concepts borrowed by the Arabs before al-Bīrūnī and by al-Bīrūnī himself.

INTRODUCTION

Al-Bīrūnī (A.D. 973-1048), the mathematician and astronomer, physicist and geographer, physician and historian, is generally held in high respect for his encyclopaedic knowledge. His mother tongue was Khwārazmin, an Iranian language and he acquired a remarkable knowledge of the Greek and Arabic sciences and literature and was in full command of the best scientific theories of his time. He stayed in India for nearly thirteen years from A.D. 1017 to A.D. 1030 and devoted himself to the study of arts, literature, and sciences of the Indians and tried to make a full use of all the documents that came to his hand, though there is evidence that many of the sources he used, have disappeared since then. However, from the works of Bīrūnī now available, we obtain a comprehensive account of his critical study of facts, extending from his reference to original sources to the analysis of scientific theories. Hence Bīrūnī's citations and observations may be considered more authoritative and valuable than those of the contemporary or even anterior Arab workers. The present paper discusses some of the concepts of arithmetic referred to by al-Bīrūnī and his contemporary Arabian scholars with reference to India.

NUMERALS

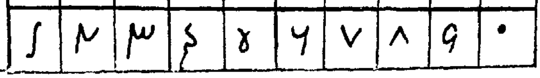
Bīrūnī wrote two books on numerals.[1] They are:

1) *Kitāb al-arqam* (Book of Numbers),

and 2) *Tazkira fi al-ḥisāb w'al-madd bi al-arqam al-Sind W'al Hind* (Description of arithmetic and system of counting with the numerals of Sind and India)

It is not known whether these two works are still extant. In his *Athār-al Bākiyā*[2] (Chronology of Ancient Nations—1000 A.D.), Bīrūnī has incidentally referred to two types of notation of numbers namely the alphabetic (*abjd*) system (*Hurūf al-jummal* or *Ḥisāb al-jummal*) and the modern numerals as *al-arqam al-Hind* (Indian numerals). Bīrūnī in his *Kītāb al-tafhim li-awā'il*

sināʹat-tanjīm[3] (The Book of Instructions in the Art of Astrology) has given both these notations of numbers. The alphabetic system (*Hurūf al-jummal*), as used by Bīrūnī was apportioned in orderly sequence to 26 alphabets though initially there was no particular order in their use. It was based on the values of the alphabets without use of decimal place-order. This notation seems to have been used exclusively by Arabian astronomers. The other system (*al-arqam al-Hind*) expressed in symbols by Bīrūnī is shown in the following table:—

Numerical symbols used by Bīrūnī (c. 1000 A.D.)	1	2	3	4	5	6	7	8	9	0
										

As regards Indian numerals Bīrūnī wrote that Vyāsa, son of Parāśara, rediscovered fifty *akṣaras* (letters) from some lost and forgotten alphabets, which were originally lesser in number. These alphabets, by and large, helped to develop various forms or systems of local alphabets through their use in different regions in India. The symbols used for these alphabets were varied in shape and character. In his *Tārikh al-Hind*[4] (Chronicles of India), Bīrūnī writes: "As in different parts of India the letters have different shapes, the numerical signs too, which are called *aṅka*, differ. The numerical signs which we use are derived from finest forms of Hindu signs".[5]

As regards the sign of zero Bīrūnī[6] writes, "when zero, has to be written in places lacking a number, its circle must have a line over it 8 touching, to distinguish from *hā*, but in the Indian notation this line is unnecessary for there is no resemblance to *hā*". The dot symbol (•) was then also used as can be seen from Bīrūnī's table. Al-Khwārizmī (*c.* A.D. 825), an eminent Arabic mathematician, wrote a work on Hindu system of numeration entitled *Algoritmi de numero Indorum* (Latin translation of the 12th century is available; the original Arabic version of the work is lost). A few other Arabian scholars besides Severus Sebokht, a Christian monk, have also referred to Hindu numerals. Hence there is no ambiguity that numerical signs for nine units and zero are derived from the Hindu signs. For this reason Reinaud, Woepcke, Rosen, Strachey, Taylor and a host of other writers came to the same conclusion.[7] George Sarton,[8] remarked that al-Bīrūnī's account gave the best medieval account of the Hindu numerals.

Decimal Place-value System

The numerals (*al-arqam al-Hind*) given in the table have been used by Bīrūnī in his book, *Kitāb al-tafhim* to express decimal place-value system or concept. He kept silent over the fact whether the concept of decimal place-value was borrowed from India or developed by him independently. However, he has admitted that while the Greeks and the Arabs did not go beyond 10^3, the Indians could easily compute much beyond that with the help of a decimal

scale. This leads evidently to the discussion whether the decimal place-value concept was originally known in India before Bīrūnī. In India it is well-known that from the time of the *Saṃhitās* (B.C. 1500), a decimal scale[9]: *eka* (1), *daśa* (10, *śata* (10^2), *sahasra* (10^3...... *upto* 10^{18} was known and a large number like, *ṣaṣṭiṃ sahasrā sapta śatāni navatiṃ nava* (60, 799) was expressed by placing the word-numerals of higher order to be followed by one of lower order in the scale. This successive placing of *sahasra, śata, daśa, eka* for the expression of a large number in left to right order of gradually decreasing values obviously shows that the idea of decimal place value was implied. From the second century A.D. onwards, this order of expressing a number by using word numerals was reversed with a new type of word numerals, for example, in the new type *netra* (2)—*aṣṭa* (8)—*śara* (5)—*rātri* (1) signifies actually 1582 instead of 2851. This is verified from examples available in Sanskrit astronomical and mathematical works and by the evidence derived from the Sanskrit text numerals as well as the symbols used for them in the Dinaya Inscription (A.D. 760). This order is so for no obvious reason, except possibly for emphasizing the idea that the decimal place-value concept remains unchanged in whatever order (right or left) the numerals are arranged when the number is always expressed by reading from higher to lower order in the scale (*vide* Table, 177-8). It is significant to note that a number was also invariably expressed by the use of symbolic numerals retained in a left to right order since their introduction in India from A.D. 400.

That the decimal place value concept was current has also been attested by Vyāsa[11] (before A.D. 700) in his philosophical work, *Yogadarśanabhāṣya* (ch. 3, *sū.* 13) thus: *'yathaikā rekhā śatasthāne śataṃ daśasthāne daśa ekaṃ caikasthāne'*, i.e. 'a (numerical) sign, denotes hundred in the *śata* place, ten in the *daśa* place, and one in the *eka* place'. The very similar language is used by Śaṇkarācārya (*c.* A.D. 800) in his *bhāṣya* on the *Vedāntasūtra* (II. 2. 17): *yathā caikāpi satī rekhā sthānānyatven niviśamānaika daśaśatasahasrādi śabdapratyayabhedamanubhavati* i.e 'one and the same (numerical) sign when occupying different places, is conceived as meaning either one, ten, one hundred, a thousand etc.' These two illustrations establish without any doubt that by the time of compositions of these two works, the decimal place-value concept must have been well established and was generally known.

Thus we see in India a continued tradition of a decimal scale system originally from the *Saṃhitā* period (*c.* B.C. 1500) followed by the use of symbols in the scale since A.D. 400 involving the idea of decimal place-value. There is no evidence, however, that such use of the word-numerals followed by symbolic use based on a decimal place-value scale, was known to any other country at a time before their use in India. This demonstrates that al-Bīrūnī obviously made use of this Indian concept of decimal place value in his book.

It is rather strange to note that al-Khwārizmī[12] (A.D. 825) who translated Indian work on arithmetic did not refer to the use of the decimal place-value in India, for he expressed 7586 as $7000+500+80+6$, indicating thereby his

TABLE

Decimal Scale			(10^5)	(10^4)	(10^3) *sahasra*	(10^2) *śata*	(10) *daśa*	(1) *eka*	*Number*
(A) Word-numerals of the *Saṃhitā* Period (B.C. 1500)	(1)	*Ṛg. Veda* (i.53.9)	—	—	*ṣaṣṭiṃ sahasrā* (60×10^3)	—	*navatiṃ* (9×10)	*nava* (9×1)	= 60, 099
	(2)	*Ṛg. Veda* (i. 164, 11)	—	—		*sapta-śatāni* (7×10^2)	*viṃśati* (2×10)	—	= 720
	(3)	*Ṛg. Veda* (ii. 1. 8)	—	—	*sahasrāni* (1×10^3)	*śata* (1×10^2)	*daśa* (1×10)	—	= 1110
Decimal Scale			(1)	(10)	(10^2)	(10^3)	(10^4)	(10^5)	
(B) Word-numerals[18] new type from c. A.D. 200	(1)	*Pauliśa Siddhānta* (A.D 200.)	*nava* (9)	*ṛtu* (6)	*rūpa* (1)	—	—	—	= 169
	(2)	*Sūrya-Siddhānta* (A.D. 400)	*ṣaṭ* (6)	*śūnya* (0)	*indriya* (5)	*nava* (9)	*vasu* (8)	*viṣaya* (5)	= 589506
	(3)	Inscription at Cambodia (A.D. 604)	*rasa* (6)	*dasra* (2)	*śara* (5)	—	—	—	= 526 (śaka)
	(4)	Inscription at Jāvā (A.D. 732)*	*śruti* (4)	*indriya* (5)	*rasa* (6)	—	—	—	= 654 (śaka)
	(5)	Dinaya Inscription at Jāva A.D(760.)	*nayana* (2)	*vasu* (8)	*rasa* (6)	—	—	—	= 682 (śaka)

TABLE

Decimal Scale			(10^6)	(10^5)	(10^4)	(10^3)	(10^2)	(10)	(1)	*Number*
(C) Numerical Symbols from A.D. 400	(1)	*Bakhshālī Ms.* (A.D. 400) 17 verso							•	= 300
	(2)	*Bakhshālī Ms.* (A.D. 400) 56 recto							•	= 846720
	(3)	Khèmre Inscription at Sambor (A.D. 683)						•		= 605 (śaka)
	(4)	Malaya Inscription at Palembang (A.D. 684)						o		= 606 (śaka)
	(5)	Dinaya Inscription at Java (A.D. 760)								= 682 (śaka)

ignorance of decimal place-value numeration. Abu al-Wafa (10th century A.D.) in his work *Mā Yahtāj ilaihi al-'Ummāl wa al-kitāb min Sinā'at al-Ḥisāb* (The Sufficient Book on Mathematics) expressed number in words without any use of numerical symbols and used a simple dash for zero. Al-Nasavī[13] c. A.D. 1030,), however, states that the symbolism of numbers was unsettled in his day. but in his work in Hindu arithmetic, *Al-Mugni'fī al-Ḥisāb al-Hindi* (the Sufficient Account on the system of Indian Mathematics) he has made use of the decimal place-value concept. It may, therefore, be concluded that the decimal place-value concept was borrowed from India by al-Nasavī or some other scholar before Bīrūnī of which the later scholars were not aware of.

PĀṬĪ AND THE USE OF DUST NUMERALS

The Indians performed their computations on a board (*pāṭī*) with a piece of chalk or on sand (*dhūli*) spread on the ground or on the *pāṭī*. Thus, terms like *pāṭīgaṇita* (the science of calculation on the board) or *dhūlikarma* (dust numerals) came to be in use in India. The latter term has been referred to by Bīrūnī[14] and others.[15] The *pāṭī* was used by Indians for computations of fundamental operations. In the methods of multiplication, division, square, cube, square-root and cube-root etc. given by Brahmagupta (A.D. 628), Mahāvīra (A.D. 850), Pṛthūdakasvāmī (A.D. 868), *Śrīdhara* (A.D. 900), Bhāskara II (A.D. 1150)[16] etc. where the figures were large and several lines could not be fitted on the *pāṭī*, the practice of obliterating digits or rubbing out digits by fingers not required for subsequent stages of work was common. For working on the *pāṭī* the student had to commit to memory all the rules required for the solution of the problems. Along with each step in the process of calculations the *sūtra* (rule) had to be repeated by the student, as is found still in a village *pāṭhaśālā* (beginners' school) in India, under strict supervision and guidance of the teacher. The details of the methods or procedures adopted for this purpose are available to us in various commentaries, viz. the commentaries of Pṛthūdaka, Gaṇeśa, on the *Brāhmasphuṭasiddhānta* and the *Līlāvatī* etc. Saidan[17] recently after a survey of nineteen important Arabic texts and from statements of al-Uqlīdisī (A.D. 952) came to a definite conclusion that the words like *takht* (dust computing board), *ghubār* (dust numerals), *turāb* (dust numerals), and *al-hind* (Indian or Indian way) used extensively by the Arab scholars got their ideas from the Indian practice when the knowledge of the Hindu arithmetic found its way to Arabia. There is a great deal of controversy regarding the actual import of the terms.[18] But our subsequent discussion on the subject will also justify the views expressed by Saidan. It is interesting to note that it is the Arabian scholar al-Uqlīdisī (A.D. 952) who first of all used paper and ink to improve the system in a better way.

The use of *pāṭī* as a type of abacus[19] has been suggested by some European scholars like Bayley, Fleet and several others. we find no reasonable ground for such a suggestion, for it is indeed difficult to trace any connection or similarity between *pāṭī* and abacus.

INDIAN ARITHMETIC IN ARABIA

There were two different arithmetical systems in vogue in Arabia, one using Hindu methods and the other, the indigenous Arabic method. This view is generally held by scholars, like Cantor, Medowi (a Russian scholar) and Saidan, deeply versed in Arabic literature.[20] In a critical survey it is found that the Arabic Arithmetic received much from that of the Indians in the method of multiplication, division, square-root, cube-root, in the concept of highest common factor and lowest common multiple and probably in some other topics. The Arabians utilised the Indian methods of calculation in a more explicit form. In the computation of fundamental operations. the scholars like al-Khwārizmī (A.D. 825), Abu al-Wafa (10th century), al-Uqlīdisī (A.D. 952), al-Nasavī (*c.* A.D. 1025) and various other scholars adopted various schemes which are Hindu in disguise. To clarify this, a reference to the working in the method of multiplication adopted by the Arabians is shown below. The method is characterized by the use of paper and ink where *pāṭī* and chalk have been used by the Indians.

To multiply 324 by 753.

(A) *Hindu Method.*[21]

(i) Two numbers are arranged thus and the multiplication starts with numbers shown by arrow mark.

$$\begin{array}{cccccl} 7 & 5 & 3 & & & = \text{multiplier} \\ & & 3 & 2 & 4 & = \text{multiplicand.} \end{array}$$

(ii) The result 3×7, i.e. 21 is placed below 7 as put here and next two numbers to be multiplied are shown by arrow-mark.

$$\begin{array}{cccccc} & 7 & 5 & 3 & & \\ 2 & 1 & & 3 & 2 & 4 \end{array}$$

(iii) The product of 5 and 3, i.e. 15 is placed in such a way that 5 of the number 15 is placed under the multiplier 5 and 1 of the number 15 is carried over to the left. Evidently 1 of the number 21 is rubbed out and $(1+1)=2$ is substituted, in its place. The multiplication then starts with numbers shown by arrow-mark.

$$\begin{array}{cccccc} & 7 & 5 & 3 & & \\ 2 & 2 & 5 & 3 & 2 & 4 \end{array}$$

(iv) Now the result 3×3, i.e. 9 is placed under the multiplier 3 giving

$$\begin{array}{cccccc} & & 7 & 5 & 3 & \\ 2 & 2 & 5 & 9 & 2 & 4 \end{array}$$

(v) The multiplier, (i.e. 753) is now moved one place to the right, and the multiplication begins with numbers joined by the mark.

$$\begin{array}{cccccc} & & 7 & 5 & 3 & \\ 2 & 2 & 5 & 9 & 2 & 4 \end{array}$$

(vi) The product of 7×2, i.e. 14 must be placed under the multiplier 7 such that 5 is rubbed out and $(5+4)=9$ is put in its place. 1 of the number 14 is carried over to the left, so that the left number 2 is replaced by $(2+1)=3$. Thus we get

$$\begin{array}{cccccc} & & 7 & 5 & 3 & \\ 2 & 3 & 9 & 9 & 2 & 4 \end{array}$$

Continuing this process upto the last digit 4 of the multiplicand, we get the result as follows:

$$\begin{array}{cccccc} & & 7 & 5 & 3 & \\ 2 & 4 & 3 & 9 & 7 & 2 \end{array}$$

∴ The result of the product is 243972.

(B) *Arabic Method*[22]:

$$\begin{array}{cccccc} & 4 & 3 & 9 & 7 & \\ & \cancel{3} & \cancel{0} & \cancel{7} & \cancel{6} & \\ & \cancel{2} & \cancel{8} & \cancel{9} & \cancel{8} & 2 \\ 2 & \cancel{1} & \cancel{5} & \cancel{8} & & \\ & & & 3 & 2 & 4 \quad = 243972 \\ & 7 & 5 & 3 & & \\ & & 7 & 5 & 3 & \\ & & & 7 & 5 & 3 \end{array}$$

The multiplication starts with numbers shown by arrow-mark and placed at the top of 7. Then the remaining numbers of the multiplicand 753 are

multiplied by the multiplier, number 3, following a carrying over process. Then the multiplicand 753 is shifted one place to the right and the process is repeated as in Hindu method.

The Arabic illustration is taken from the work of al-Nasavī (*c.* A.D. 1025) who calls this method *al-amal al-Hindi* and *tārikh al-Hindi* (the method of the Hindus).

Al-Uqlīdisī (A.D. 952) treated the method of extraction of cube-root at the end of his book, *Kitāb al-Fuṣūl fi al-Ḥisāb al-Hindi* (the Book having different chapters on Indian Mathematics). He pointed out that the previous scholars had no satisfactory knowledge of cube-root and he himself was the first to deal with the operation in full. Bīrūnī was also aware of this besides other fundamental operations. In this connection it must be noted that the method of square-root and cube-root based on a decimal place-value scale were already discovered by Āryabhaṭa I (A.D. *c.* 499) in India. The algebraical methods of extracting square-root and cube-root were also known to the Greeks who had no knowledge of decimal place-value. In view of the decimal and other Indian methods of calculations adopted by the Arabs, it may be presumed that the Arabic methods of extraction of square-root and cube-root were also borrowed from the Indians.

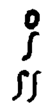
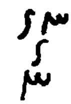
Rule of Three

The Arabs held the method of the Rule of Three (*trairāśika*) in a very high esteem as is evidenced from Bīrūnī's references. Bīrūnī himself wrote a separate treatise entitled *Fi raśikat al-Hind*[23] (The *rāśika* of the Hindus) dealing with Hindu Rules of Three. He had also used an example of *vyasta trairāśika* (the Inverse Rule of Three) in his *India.*[24] It is not yet known whether it is Bīrūnī who first introduced the Rule of Three into Arabia or was discovered by any other scholar before him. However, this rule in detail was found in the *Bakhśhālī Ms.* (4th century A.D.) and in the works of Āryabhaṭa I (A.D. 499), Brahmagupta, (A.D. 628), *Mahāvīra* (A.D. 850) etc. and was perfected before it was known to China and other parts of the world. This rule was transmitted to Arab probably in the eighth century and thence travelled to Europe where it came to be known as the Golden Rule.[25]

Fraction

Another important concept that found its way to Arabia was the use of the notation of common fraction free from linguistic and metrological limitations. Like vedic Indians[26] the Arabs used one single word-name for each of the principal fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ upto $\frac{1}{10}$. The system is basically concerned with the naming of fractions in terms of the principal ones. For $\frac{1}{12}$ is named as $\frac{1}{2}$ $\frac{1}{6}$ or $\frac{1}{3}$ $\frac{1}{4}$. Another practical system was to avoid this by expressing fractions in terms of submultiples of certain metrological unit, namely *dīrham* and *qīraṭ*. The study remained linguistic with no use of symbolic forms. The symbolic expression of fraction is first found in India in the *Bakhshālī Ms*[27].

(A.D.400). Here $\begin{matrix}2\\1\\2\end{matrix}$ represents $2\frac{1}{2}$, $\begin{matrix}1\\1\\2\end{matrix}$ as $1\frac{1}{2}$ and there are given many other similar examples. Taylor[28] who examined a number of Mss. of the *Līlāvatī* and its commentaries also informs us that this was the manner of expressing fraction of the Hindus. The Arabian scholar al-Nasavī[29] (A.D. 1034) first used $\begin{matrix}237\\8\\12\end{matrix}$ to mean $237\frac{8}{12}$. Examples of similar types are also found to appear in eleventh century Arabian works. For instance.

$\frac{1}{2}$ is written as , $\frac{1}{11}$ as , $13\frac{1}{3}$ as  etc.

Bīrūnī referred to the operations of fraction and the reduction of fractions without any remarks on Indian or Arabian system. From above, there is no denying the fact that Arabian knowledge of fraction was derived from that of the Indians.

To sum up, the paper gives us a glimpse of the type of a sample survey to draw a picture of the nature of Indian arithmetic as handed down to the Arabs. To find out the actual elements and the concepts borrowed by the Arabians, the Arabic works between A.D. 800 to A.D. 1200 must be thoroughly explored.

Acknowledgements

I am grateful to Sri S. N. Sen and Dr. Pulin Bihari Chakravarti for their kind interests in preparation of the paper.

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